Design Calculations & Real Behaviour (Hambly's Paradox)

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1. Introduction

The present work is meant to highlight the idea that engineering design calculations (lately most of the time using computer design software), cannot always quantify real loading conditions. For this I will use the four-legged stool example of past ICE president Edmund Hambly, known also as Hambly's paradox.

The paradox

Q_1: A man weighing 600N sits on a three-legged stool. What basic force should each leg be designed for?

A_1: The stool is supposed to be symmetrical, the man sits in the centre of the seat, so the answer is of course, that each leg should be designed to carry a force of 200N.

Q_2: The same man now sits on a square stool with four legs, one at each corner, the stool and the loading are symmetrical. What force should each leg of the stool be designed for?

A_2: The apparent trivial answer of 150N is wrong, and this is what the paradox is about. The legs of the stool should be designed for a greater force then the three-legged stool.

2. Explanation

One of the structural concepts that leads to the understanding of the paradox is that:

1. ***A plan is determined by three points.***

So, in the first case all three legs will be in contact with the floor, supporting the weight of the man.

The second reason that leads to the paradox is:

2. ***In practice, we can never create perfect/ideal elements as often used in design.***

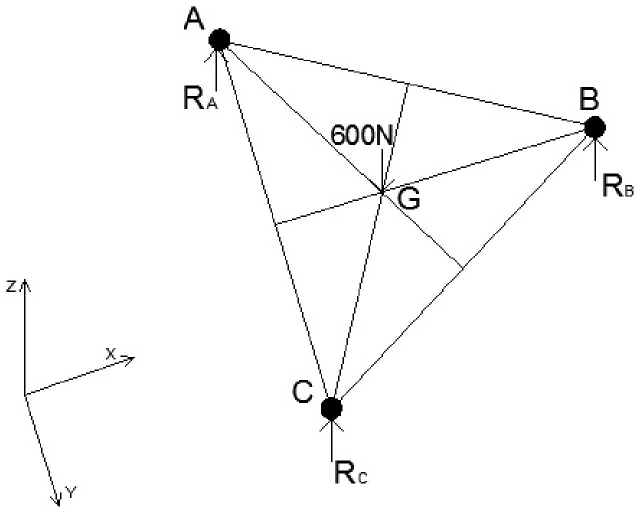
Therefore, if one of the four legs is shorter by only a fraction of a millimeter, the other three legs will be in contact with the floor and will support the weight of the man. Of course a lot of other execution defects will lead to the same effect, like: the floor is not perfectly plane, or some of the legs having some previous deformations. In the end, all this lead to the fact that one leg is not in contact with the floor, so certainly the force it is carrying is zero. By simple statics, the force in the leg diagonally to this one will also be zero, even if it is in contact with the floor (Moment equilibrium about two horizontal axis, X, Y). This leads to the fact that only 2 legs carry the force of 600N, so each should be designed for 300N>200N.

3. Design process

3.1 In design analysis we first find the values of external forces, in this case **F=600N**.

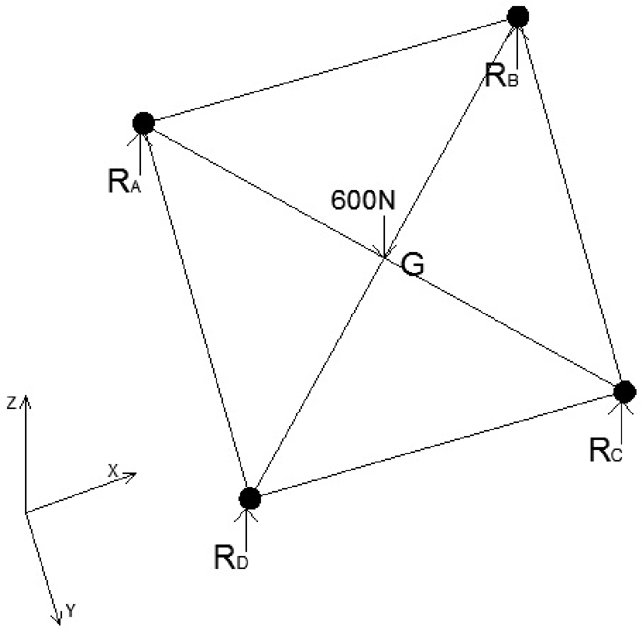
3.2 After, we want to determine the internal stress resultants, in this case **Axial forces**.
In this simple case, the axial forces are equal to the reactions from the floor. **N=R**

3.3 Three-legged stool



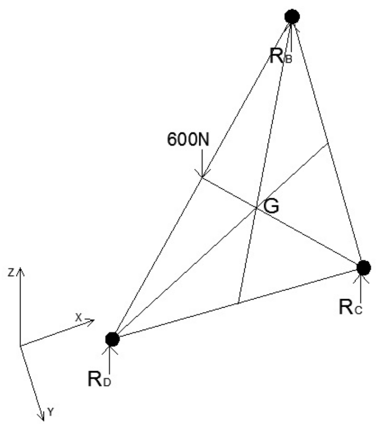
We can write 3 equations of overall equilibrium ($\Sigma F_Z=0$, $\Sigma M_X=0$, $\Sigma M_Y=0$), and we have 3 unknown reactions (R_A, R_B, R_C), so the structure is statically determinate. The moment equilibrium equations tell us that $R_A=R_C$, $2R_B=R_A+R_C$ => $R_A=R_B=R_C$, and the force equilibrium equation that $R_A+R_B+R_C=600N$, so $R_A=R_B=R_C=200N$.

3.4 Four-legged stool



In this case we can write only the same 3 overall equilibrium equations as before, but we have 4 unknown reactions so the structure is statically indeterminate. Although, we can tell that $R_A+R_B+R_C+R_D=600N$, $R_A=R_C$, $R_B=R_D$ and $R_A,R_B,R_C,R_D \leq 300N$.

Let us assume that the leg from point A is the one shorter, so $R_A=0$. The problem transforms into:



The problem is reduced to a three legged stool, only the force is now not applied in the center of the slab of the stool. As we said before, from moment equilibrium equations we can determine that R_C=0. We can confirm this by the fact that the 600N force has the application point on the line between point B and D in a triangle, so only R_D and R_B will take the force => R_C=0. So we now proved that R_B=R_D= 300N, this being the force for which all legs should be designed, not knowing which of them will be shorter.

1.5 Elastic analysis

All the analysis and the assumptions made so far were regarding elastic behavior of the materials. Besides finding reactions, design involves also: **defining material properties** and setting **geometrical conditions** such as **deformations compatibility**.

The result of designing for 300N is considering a perfect linear-elastic material, so the this is an **ELASTIC SOLUTION**.

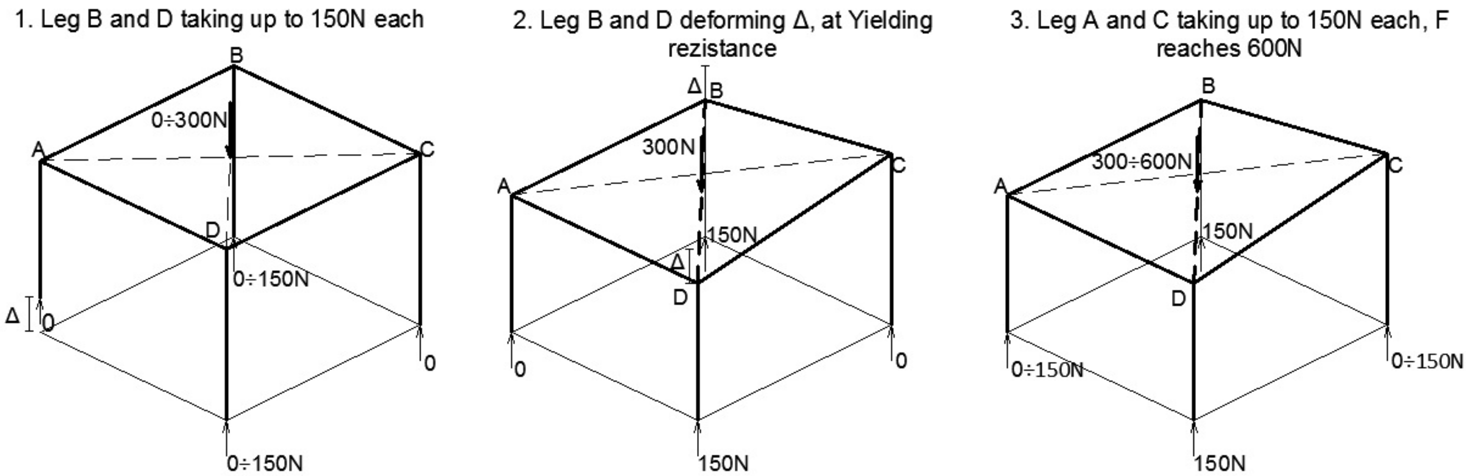
1.6 Plastic analysis

Elastic analysis for the four-legged stool is very complicated because the structure is statically indeterminate. This is because the real boundary conditions cannot be determined. If no such conditions are taken into consideration, the result will be 150N reactions, which are obviously wrong.

Now plastic method was developed by the Steel Structures Research Committee in 1930's, by running tests on steel-framed buildings, regarding the fact that elastic-analysis is not covering all aspects of structural behavior.

The plastic method involves designing the legs of the stool to have a **ductile failure**. This means that the 2 legs which take the whole load at the beginning will reach yielding resistance, let us say at 150N when plastic hinges are formed. This will allow this 2 legs to plastically deform, making the 4^{th} leg to come in contact with the floor.

This involves a quasi-static process, that means the 600N force will be slowly increased from 0 to 600N.



The thing is that the legs reaching their yielding capacity (B,D) should be able to sustain their squash load without becoming unstable. Stability should be perfectly assured and the legs should not fail from buckling. If one of these will occur, the whole structure will fail.

4. Building our own 3 and 4 legged stools

Now we know the theory:

- 3-legged designed for 200N
- 4-legged designed Elastic for 300N
- 4-legged designed Plastic for 150N

Let us build our own 3;4 legged structures to see if Dr. Hembly's paradox really happens. The model will have a lot of "problems" regarding the difference between design and reality, due to the fact it is homemade. But that is the whole purpose, this will amplify the very-small imperfections that exist in probably each and every structure. Although aware of this, I will try to build the model as close as it can get to the ideal one.

4.1 The Legs

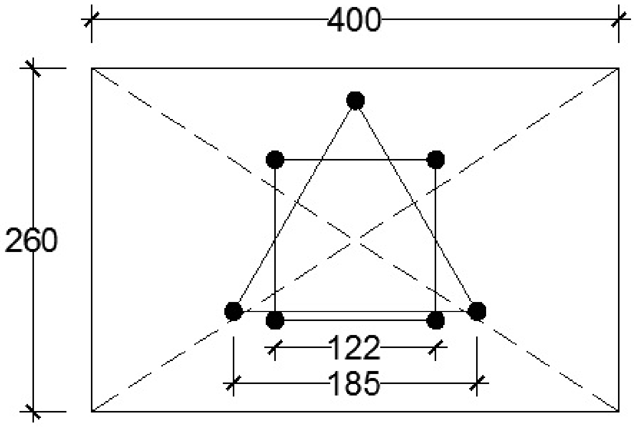
The material used for legs is A4 usual paper, they will have a tubular cross-section, with 1 stiffener at the middle made of rubber-band. I'll be aware of:

- Same cross-section diameter
- Stiffened right at the middle
- Plane cross-sections at the ends

4.2 The Slab

The slab I will use has to be rigid, not to suffer any deformations, so it can transmit the load equally to the legs, without failing reaching bending capacity. I will use a metal rectangular tray. I'll be aware of:

Marking the center of the tray on both faces. One for applying the load symmetrically and the other for placing the legs symmetrically. It is heavy compared to the paper so this is also important for transmitting the self-weight load.



Both arrangements of legs (square, triangle) and the slab have the center in the same point. The area determined by the triangle and square are designed to be the same ($A_S=A_T$):

$L_S^2 = (L_T \cdot \frac{\sqrt{3}}{2} L_T) / 2$; $122 \cdot 122 = 34225 \cdot 1.73 / 4$; $14884 \approx 14802$ mm^2.

4.3 The Load

I will load the structure with bottled water. I am aware that tubular sections are very effective in taking compression because of the high gyration radius, so probably I will need a lot of water. Regarding this I'll be aware of:

- Using an improvised beaker (1/2L measured), to control the loading increase
- Determine the center of gravity for the bottles used (correlation between them and with the slab's). From left to right:

1. Used as beaker: adding 1/2L; **2.**5L cap.; **3.**2L cap; **4.**2L cap.

I will fill all bottles with different quantities, adding 0.5L, so a bottle may have different weights for different tries.

The way we approach the problem is the other way around, meaning we consider all the legs having the same designed capacity (R), and try to find the load they can take.

5. Testing

5.1 Three-legged

After trying the structures before couple of times to see approximately how much load they can take, I will start with F=4L, being sure that neither of them fail at this load.


N=0


N=4L


N=4+0.5=4.5L


N=4+1=5L


N=4+1.5=5.5L


N=4+2=6L


N=4+2+0.5=6.5L


N=4+2+1=7L


N=5+1.5+1=7.5L


N=5+1.5+1.5=8L


FAILING: N=5+2+1.5=8.5L

- Deformed shape of the legs is not very relevant because they suffered deformations after failing, when the structure collapsed.
- Although, we can observe that the structured failed because of buckling at the top part of the left leg.
- Collapse happened sudden, without warning
- Hard to observe any plastic deformations
- **Load carrying capacity = 8L**.

5.2 Four-legged

N=0

N=4L

N=4+0.5=4.5L

N=4+1=5L

N=4+1+0.5=5.5L

N=4+2=6L

N=4+2+0.5=6.5L

N=4+2+1=7L

FAILED: N=4+2+1.5=7.5L

- When the whole structure collapsed the paper legs where a lot affected so their shape after failing is not very relevant.
- Although, we can observe that the 2 legs from the left are more deformed, especially at the top side. These are the two diagonal legs who took most of the load, and they suffered big local deformations at the top.
- The collapse happened suddenly, all at once, because of the brittle behavior of paper.
- **Load carrying capacity = 7L**.

6. Conclusion

1. Three-legged stool capacity (8L) > Four-legged stool capacity (7L). **Paradox confirmed.**
2. Using paper as the material for legs helped the paradox happen, because it has a brittle behavior. Therefore we can say it was an elastic experiment.
3. To calculate the actual behavior of a structure you have to take into consideration all three structural statements: **EQUILIBRIUM, MATERIAL PROPERTIES, DEFORMATION COMPATIBILITIES**.
4. Calculations alone, do not always lead to the actual state of a structure!

REFFERENCES:

- "Proc. Institute of Civil Engineers", 114, Nov., pg.161-166. Paper 11082 http://www.icevirtuallibrary.com/content/article/10.1680/icien.1996.28903
All information used was from above Paper.
First page background picture took from the same Paper.
Topic inspired from Dr. Parthasarathi Mandal, "Elastic and Plastic analysis of structures" first lecture.